

# Gauge dependence in the theory of non-linear spacetime perturbations

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## Abstract

Diffeomorphism freedom induces a gauge dependence in the theory of space-time perturbations. We derive a compact formula for gauge transformations of perturbations of arbitrary order. To this end, we develop the theory of Taylor expansions for one-parameter families (not necessarily groups) of diffeomorphisms. First, we introduce the notion of knight diffeomorphism, that generalises the usual concept of flow, and prove a Taylor's formula for the action of a knight on a general tensor field. Then, we show that any one-parameter family of diffeomorphisms can be approximated by a family of suitable knights. Since in perturbation theory the gauge freedom is given by a one-parameter family of diffeomorphisms, the expansion of knights is used to derive our transformation formula. The problem of gauge dependence is a purely kinematical one, therefore our treatment is valid not only in general relativity, but in any spacetime theory.

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# 1 Introduction

In the theory of spacetime perturbations [1, 2, 3], one usually deals with a family of spacetime models  $M_\lambda := (\mathcal{M}, \{T_\lambda\})$ , where  $\mathcal{M}$  is a manifold that accounts for the topological and differential properties of spacetime, and  $\{T_\lambda\}$  is a set of fields on  $\mathcal{M}$ , representing its geometrical and physical content. The numerical parameter  $\lambda$  that labels the various members of the family gives an indication of the ‘size’ of the perturbations, regarded as deviations of  $M_\lambda$  from a background model  $M_0$ . Perturbations are described as additional fields in the background, defined as  $\Delta T_\lambda^\varphi := \varphi_\lambda^* T_\lambda - T_0$ , where  $\varphi_\lambda : \mathcal{M} \rightarrow \mathcal{M}$  is a diffeomorphism that provides a pairwise identification between points of the perturbed spacetime and of the background, and  $\varphi_\lambda^*$  denotes the pull-back. Of course, such an identification is arbitrary, and this leads to a gauge freedom in the definition of perturbations. Under a change  $\varphi_\lambda \rightarrow \psi_\lambda$  of the point identification mapping, a perturbation transforms as  $\Delta T_\lambda^\varphi \rightarrow \Delta T_\lambda^\psi$ , with

$$\Delta T_\lambda^\psi = \Phi_\lambda^* \Delta T_\lambda^\varphi + (\Phi_\lambda^* T_0 - T_0) , \quad (1.1)$$

where  $\Phi_\lambda := \varphi_\lambda^{-1} \circ \psi_\lambda$  is a diffeomorphism on  $\mathcal{M}$ .

In the perturbative approach, one tries to approximate  $T_\lambda$  expressing  $\Delta T_\lambda^\varphi$  as a series,

$$\Delta T_\lambda^\varphi = \sum_{k=1}^{n-1} \frac{\lambda^k}{k!} \delta^k T^\varphi + O(\lambda^n) , \quad (1.2)$$

where  $n$  is the order of differentiability with respect to  $\lambda$  of  $\Delta T_\lambda^\varphi$ , and then solving iteratively the field equations for the various terms  $\delta^k T^\varphi$ . It is then important to know how the latter transform under a change of gauge. Until very recently, only the first order terms,  $\delta^1 T^\varphi$ , have been considered; in this case, it is well-known that the representations of a perturbation in two different gauges differ just by a Lie derivative of the background quantity  $T_0$  [1]. However, non-linear perturbations are now becoming a valuable tool of investigation in black hole and gravitational wave physics [4], as well as in cosmology [5]. Their behaviour under gauge transformations can be derived by Taylor-expanding (1.1) with respect to  $\lambda$ .

This apparently straightforward procedure presents a difficulty, though. Even if one chooses, as usual, point identification maps that are one-parameter groups with respect to  $\lambda$ , the family of diffeomorphisms  $\Phi_\lambda$  is *not* a one-parameter group [3], i.e., it does not correspond to a flow on  $\mathcal{M}$ . While flows on manifolds are well understood and widely discussed in the literature, more general one-parameter families of diffeomorphisms are not. Only some fragmentary statements about them can be found in a few papers [6, 7, 8, 9]. Therefore, in order to extract from (1.1) the relationship between  $\delta^k T^\varphi$  and  $\delta^k T^\psi$ , one must first develop the theory of Taylor expansions for general one-parameter *families* of diffeomorphisms, not necessarily forming a local group.

The purpose of the present article is to provide the mathematical framework needed for this purpose. Roughly, the discussion generalises section 2 of reference [3] from the analytic to the  $C^n$  case, but we also derive here a compact formula that gives directly the gauge transformation to an arbitrary order  $k$ . The paper is organised as follows. In the next section we define particular combinations of flows that we dub knight diffeomorphisms, and present our main result (Theorem 1). This establishes that arbitrary one-parameter families of diffeomorphisms can be approximated by families of knights, so that all one needs is a suitable expression for the Taylor expansion of knights, which is derived in section 3. Then, Theorem 1 is proved in section 4. Section 5 contains the application to (1.1), i.e., our formula (5.1) and some concluding remarks.

In the following, we shall work on a finite-dimensional manifold  $\mathcal{M}$ , smooth enough for all the statements below to make sense. In order to avoid cumbersome talking about neighbourhoods, we shall often suppose that maps are globally defined. This assumption simplifies the discussion, without altering the results significantly. Also, we specify the class of differentiability of an object only when it is really needed. Finally, let us recall that a one-parameter family of diffeomorphisms of  $\mathcal{M}$  is a differentiable mapping  $\Phi : \mathcal{D} \rightarrow \mathcal{M}$ , with  $\mathcal{D}$  an open subset of  $\mathbb{R} \times \mathcal{M}$  containing  $\{0\} \times \mathcal{M}$ , and  $\Phi(0, p) = p$ ,  $\forall p \in \mathcal{M}$ . As we have already been doing, we shall write, following the common usage,  $\Phi_\lambda(p) := \Phi(\lambda, p)$ , for any  $(\lambda, p) \in \mathcal{D}$ .

## 2 Knight diffeomorphisms

Let  $\phi^{(1)} : \mathcal{D}_1 \rightarrow \mathcal{M}, \dots, \phi^{(k)} : \mathcal{D}_k \rightarrow \mathcal{M}$  be flows on  $\mathcal{M}$ , generated by the vector fields  $\xi_1, \dots, \xi_k$ , respectively. We can combine  $\phi^{(1)}, \dots, \phi^{(k)}$  to define a new one-parameter family of diffeomorphisms  $\Psi : \mathcal{D} \rightarrow \mathcal{M}$ , with  $\mathcal{D}$  a suitable open subset of  $\mathbb{R} \times \mathcal{M}$  containing  $\{0\} \times \mathcal{M}$ , whose action is given by

$$\Psi_\lambda := \phi_{\lambda^k/k!}^{(k)} \circ \dots \circ \phi_{\lambda^2/2}^{(2)} \circ \phi_\lambda^{(1)}. \quad (2.1)$$

Thus,  $\Psi_\lambda$  displaces a point of  $\mathcal{M}$  a parameter interval  $\lambda$  along the integral curve of  $\xi_1$ , then an interval  $\lambda^2/2$  along the integral curve of  $\xi_2$ , and so on (see Fig. 1 for the case  $k = 2$ ). For this reason, we shall call  $\Psi_\lambda$ , with a chess-inspired terminology, a *knight diffeomorphism of rank  $k$*  or, more shortly, a *knight*. The vector fields  $\xi_1, \dots, \xi_k$  will be called the *generators* of  $\Psi$ .

The utility of knights stems from the fact that any  $C^n$  one-parameter family  $\Phi$  of diffeomorphisms can always be approximated by a family  $\Psi$  of knights of rank  $n - 1$ , as shown by the following

**Theorem 1** *Let  $\Phi : \mathcal{D} \rightarrow \mathcal{M}$  be a  $C^n$  one-parameter family of diffeomorphisms. Then  $\exists \phi^{(1)}, \dots, \phi^{(n-1)}$ , flows on  $\mathcal{M}$  such that, up to the order  $\lambda^n$ , the action of  $\Phi_\lambda$  is equivalent*

to the one of the  $C^n$  knight

$$\Psi_\lambda = \phi_{\lambda^{n-1}/(n-1)!}^{(n-1)} \circ \cdots \circ \phi_{\lambda^2/2}^{(2)} \circ \phi_\lambda^{(1)} . \quad (2.2)$$

This result allows one to use knights in order to investigate many properties of arbitrary diffeomorphisms. In a sense, knights play among the one-parameter families of diffeomorphisms of  $\mathcal{M}$  the same crucial role that polynomials play for functions of a real variable. We postpone the proof of Theorem 1 to section 4, after we have established some preliminary results.

### 3 Taylor expansion of flows and knights

It is easy to generalise the usual Taylor's expansions on  $\mathbb{R}^m$  [10] to the case of a flow acting on a manifold:

**Proposition 2** *Let  $\phi : \mathcal{D} \rightarrow \mathcal{M}$  be a flow generated by the vector field  $\xi$ , and  $T$  a tensor field such that  $\phi_\lambda^* T$  is a (tensor-valued) function of  $\lambda$  of class  $C^n$ . Then,  $\phi_\lambda^* T$  can be expanded around  $\lambda = 0$  as*

$$\phi_\lambda^* T = \sum_{l=0}^{n-1} \frac{\lambda^l}{l!} \mathcal{L}_\xi^l T + \lambda^n R_\lambda^{(n)} T , \quad (3.1)$$

where  $\mathcal{L}_\xi$  is the Lie derivative along the flow  $\phi$ , and  $R_\lambda^{(n)}$  is a linear map whose action on  $T$  is given by

$$R_\lambda^{(n)} T = \frac{1}{(n-1)!} \int_0^1 dt (1-t)^{n-1} \mathcal{L}_\xi^n \phi_{t\lambda}^* T . \quad (3.2)$$

This proposition has the important consequence that, for a tensor field  $T$  and a flow  $\phi$  such that  $\phi_\lambda^* T$  is  $C^n$ , one can approximate  $\phi_\lambda^* T$ , to order  $n-1$ , by the polynomial

$$\sum_{l=0}^{n-1} \frac{\lambda^l}{l!} \mathcal{L}_\xi^l T .$$

This follows from the property

$$\lim_{\lambda \rightarrow 0} R_\lambda^{(n)} T = \frac{1}{n!} \mathcal{L}_\xi^n T , \quad (3.3)$$

which implies that, for  $\lambda \rightarrow 0$ , the remainder  $\lambda^n R_\lambda^{(n)} T$  is  $O(\lambda^n)$ .<sup>1</sup>

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<sup>1</sup>Actually, this result holds also for the weaker case in which  $\phi_\lambda^* T$  is  $C^{n-}$  (i.e., it is of class  $C^{n-1}$  with a locally Lipschitzian  $(n-1)$ th derivative). However, under these conditions one does not have an explicit expression, like (3.2), for the remainder.

The proof of Proposition 2 is rather straightforward and can be omitted. We only wish to point out that it relies heavily on the property that  $\phi_\lambda$  forms a one-parameter group:  $\phi_{\sigma+\lambda} = \phi_\sigma \circ \phi_\lambda$ . It is evident from (2.1) that for knights one has, in general,  $\Psi_\sigma \circ \Psi_\lambda \neq \Psi_{\sigma+\lambda}$ , and  $\Psi_\lambda^{-1} \neq \Psi_{-\lambda}$ . Thus, equation (3.1) cannot be applied if we want to expand in  $\lambda$  the pull-back  $\Psi_\lambda^* T$  of a tensor field  $T$  defined on  $\mathcal{M}$ . The ultimate reason for this, is that a family of knights does not form a group, except under very special conditions, as shown by the following

**Theorem 3** *Let  $\Psi : \mathcal{D} \rightarrow \mathcal{M}$  be a family of knight diffeomorphisms of rank  $k$ , with generators  $\xi_1, \dots, \xi_k$ .  $\Psi$  forms a group iff there exists a vector field  $\xi$ , and numerical coefficients  $\alpha_l$ , with  $1 \leq l \leq k$ , such that  $\xi_l = \alpha_l \xi$ ,  $\forall l$ . In this case, under the reparametrisation  $\lambda \rightarrow \bar{\lambda} := f(\lambda)$ , with*

$$f(\lambda) := \sum_{l=1}^k \alpha_l \lambda^l / l! , \quad (3.4)$$

*$\Psi$  reduces to a flow in the canonical form.*

*Proof.* Let us first show that  $\xi_l = \alpha_l \xi$  is a sufficient condition for  $\Psi$  to form a group. Let  $\phi$  be the flow generated by  $\xi$ . Then  $\phi_\sigma^{(l)} = \phi_{\alpha_l \sigma}$ , and we have  $\Psi_\lambda = \phi_{\alpha_k \lambda^k / k!} \circ \dots \circ \phi_{\alpha_1 \lambda} = \phi_{\bar{\lambda}}$ . Thus, (i)  $\Psi_\sigma \circ \Psi_\lambda = \phi_{\bar{\sigma}} \circ \phi_{\bar{\lambda}} = \phi_{\bar{\sigma} + \bar{\lambda}} = \phi_{\bar{\tau}} = \Psi_\tau$ , with  $\tau = f^{-1}(\bar{\sigma} + \bar{\lambda})$ , and (ii)  $\Psi_\lambda^{-1} = \phi_{\bar{\lambda}}^{-1} = \phi_{-\bar{\lambda}} = \Psi_\rho$ , with  $\rho = f^{-1}(-\bar{\lambda})$ .

To prove the reverse implication, let us suppose that  $\Psi$  form a group. Let  $p$  be an arbitrary point of  $\mathcal{M}$ , and define the set  $\mathcal{C}_p := \{\Psi_\lambda(p) | \lambda \in I_p\} \subset \mathcal{M}$ , where  $I_p \ni 0$  is an open interval of  $\mathbb{R}$  such that  $I_p \times \{p\} \subset \mathcal{D}$ . Obviously,  $\mathcal{C}_p$  is a one-dimensional submanifold of  $\mathcal{M}$  (to see this, it is sufficient to consider a chart on  $\mathcal{C}_p$  where  $\lambda$  itself is the coordinate). Let us now consider another arbitrary point  $q \in \mathcal{C}_p$ , and ask whether it is possible that  $\mathcal{C}_q \neq \mathcal{C}_p$ . If it were so, there would be some  $\sigma \in I_q$  such that  $\Psi_\sigma(q) \neq \Psi_\tau(p)$ ,  $\forall \tau \in I_p$ . But since  $q = \Psi_\lambda(p)$ , for some  $\lambda \in I_p$ , and  $p$  is arbitrary, this would mean that, for some  $\lambda$  and  $\sigma$ , one cannot find a  $\tau$  such that  $\Psi_\sigma \circ \Psi_\lambda = \Psi_\tau$ , which would contradict the hypothesis that  $\Psi$  forms a group. Thus, each point of  $\mathcal{M}$  belongs to one, and only one, one-dimensional submanifold constructed using  $\Psi$  as above. The set of these submanifolds becomes a congruence of curves simply by suitably parametrising them; this, in turn, defines a flow  $\phi$  and a vector field  $\xi$ . Thus, if  $\Psi$  forms a group, it can be written as  $\Psi_\lambda = \phi_{\bar{\lambda}}$ , for some suitable parameter  $\bar{\lambda}$ .

In the particular case of a knight, this condition can be rewritten, using (2.1), as

$$\phi_\lambda^{(1)} \circ \phi_{-\bar{\lambda}} = \phi_{-\lambda^2/2}^{(2)} \circ \dots \circ \phi_{-\lambda^k/k!}^{(k)} . \quad (3.5)$$

Assuming  $\phi$  and the various  $\phi^{(l)}$  to be at least of class  $C^{2-}$  (which is a natural requirement, if one wants them to be uniquely determined by the respective vector fields), we

can apply (3.1) to (3.5) and get, for an arbitrary tensor field  $T$ ,

$$\left( \mathcal{L}_{\xi_1} - \frac{\bar{\lambda}}{\lambda} \mathcal{L}_{\xi} \right) T = O(\lambda) . \quad (3.6)$$

This implies that  $\exists \alpha_1 \in \mathbb{R}$  such that  $\bar{\lambda} = \alpha_1 \lambda + f_2(\lambda)$ , with  $f_2(\lambda) = O(\lambda^2)$ , together with  $\xi_1 = \alpha_1 \xi$ . Substituting into (3.5) and applying again (3.1), we find

$$\left( \mathcal{L}_{\xi_2} - \frac{f_2(\lambda)}{\lambda^2/2} \mathcal{L}_{\xi} \right) T = O(\lambda) . \quad (3.7)$$

Thus, we have also that  $\exists \alpha_2 \in \mathbb{R}$  such that  $f_2(\lambda) = \alpha_2 \lambda^2/2 + f_3(\lambda)$ , with  $f_3(\lambda) = O(\lambda^3)$ , and  $\xi_2 = \alpha_2 \xi$ . Iterating this procedure, one shows that  $\xi_l = \alpha_l \xi$ ,  $\forall l \leq k$ .  $\square$

It is clear from the proof given above that the failure of  $\Psi$  to form a group is also related to the following circumstance. For any  $p \in \mathcal{M}$ , one can define a curve  $u_p : I_p \rightarrow \mathcal{M}$  by  $u_p(\lambda) := \Psi_\lambda(p)$ . However, these curves do not form a congruence on  $\mathcal{M}$ . For the point  $u_p(\lambda)$ , say, belongs not only to the image of the curve  $u_p$ , but also to the one of  $u_{u_p(\lambda)}$ , which differs from  $u_p$  when at least one of the  $\xi_l$  is not collinear with  $\xi_1$ , since  $u_{u_p(\lambda)}(\sigma) = \Psi_\sigma \circ \Psi_\lambda(p) \neq \Psi_{\lambda+\sigma}(p) = u_p(\lambda + \sigma)$ . Thus, the fundamental property of a congruence, that each point of  $\mathcal{M}$  lies on the image of one, and only one, curve, is violated.

Let us now turn to the problem of Taylor-expanding  $\Psi_\lambda^* T$ . Although (3.1) cannot be used straightforwardly for this purpose, one can apply it repeatedly to  $\Psi_\lambda^* T = \phi_\lambda^{(1)*} \phi_{\lambda^2/2}^{(2)*} \cdots \phi_{\lambda^k/k!}^{(k)*} T$ , and get the following

**Proposition 4** *Let  $\Psi$  be a one-parameter family of knight diffeomorphisms of rank  $k$ , and  $T$  a tensor field such that  $\Psi_\lambda^* T$  is of class  $C^n$ . Then  $\Psi_\lambda^* T$  can be expanded around  $\lambda = 0$  as*

$$\Psi_\lambda^* T = \sum_{l=0}^{n-1} \frac{\lambda^l}{l!} \sum_{J_l} \frac{l!}{2^{j_2} \cdots n!^{j_n} j_1! j_2! \cdots j_n!} \mathcal{L}_{\xi_1}^{j_1} \cdots \mathcal{L}_{\xi_n}^{j_n} T + \lambda^n R_\lambda^{(n)} T , \quad (3.8)$$

where  $J_l := \{(j_1, \dots, j_n) \in \mathbb{N}^n \mid \sum_{i=1}^n i j_i = l\}$  defines the set of indices over which one has to sum in order to obtain the  $l$ -th order term, and  $R_\lambda^{(n)} T$  is a remainder with a finite limit as  $\lambda \rightarrow 0$ .

The geometrical meaning of (3.8) is particularly clear in a chart. Let us consider the special case in which the tensor  $T$  is just one of the coordinate functions on  $\mathcal{M}$ ,  $x^\mu$ . We have then, since  $\Psi_\lambda^* x^\mu(p) = x^\mu(\Psi_\lambda(p))$ , the action of an ‘infinitesimal point transformation,’ that reads, to second order in  $\lambda$ ,

$$\tilde{x}^\mu = x^\mu + \lambda \xi_1^\mu + \frac{\lambda^2}{2} \left( \xi_{1,\nu}^\mu \xi_1^\nu + \xi_2^\mu \right) + O(\lambda^3) , \quad (3.9)$$

where we have denoted  $x^\mu(p)$  simply by  $x^\mu$ , and  $x^\mu(\Psi_\lambda(p))$  by  $\tilde{x}^\mu$ . Equation (3.9) is represented pictorially in Fig. 2. The effect of  $\phi^{(2)}$  (and of higher order  $\phi$ 's) is to correct the action of the simple flow  $\phi^{(1)}$ .

Finally, let us notice that since each element of  $J_l$  has  $j_i \equiv 0, \forall i > l$ , the sum on the right hand side of (3.8) only involves the Lie derivatives along the vectors  $\xi_l$  with  $l \leq n-1$ . Thus, as far as Taylor expansions are concerned, only knights of rank lower than their degree of differentiability are really relevant.

## 4 Proof of Theorem 1

If  $\varphi$  and  $\psi$  are two diffeomorphisms of  $\mathcal{M}$  such that  $\varphi^*f = \psi^*f$  for every function  $f$ , it follows that  $\varphi \equiv \psi$ , as it is easy to see in a chart. Thus, in order to show that a family of knights  $\Psi$  approximates any one-parameter family of diffeomorphisms  $\Phi$  up to the  $n$ -th order, it is sufficient to prove that  $\Psi_\lambda^*f$  and  $\Phi_\lambda^*f$  differ by a function that is  $O(\lambda^n)$ ,  $\forall f$ . Let us therefore consider the action of  $\Phi_\lambda$  on an arbitrary sufficiently smooth function  $f : \mathcal{M} \rightarrow \mathbb{R}$ . The Taylor expansion of  $\Phi_\lambda^*f$  gives [10]

$$\Phi_\lambda^*f = \sum_{l=0}^{n-1} \frac{\lambda^l}{l!} \left. \frac{d^l}{d\lambda^l} \right|_0 \Phi_\lambda^*f + \lambda^n R_\lambda^{(n)}f, \quad (4.1)$$

with

$$R_\lambda^{(n)}f = \frac{1}{(n-1)!} \int_0^1 dt (1-t)^{n-1} \left. \frac{d^n}{d\lambda^n} \right|_{t\lambda} \Phi_\lambda^*f. \quad (4.2)$$

Let us define  $n-1$  linear differential operators  $\mathcal{L}_1, \dots, \mathcal{L}_{n-1}$  through the recursive formula

$$\mathcal{L}_l f := \left. \frac{d^l}{d\lambda^l} \right|_0 \Phi_\lambda^*f - \sum_{J'_l} \frac{l!}{2^{j_2} \dots (l-1)! j_{l-1}! j_1! j_2! \dots j_{l-1}!} \mathcal{L}_1^{j_1} \mathcal{L}_2^{j_2} \dots \mathcal{L}_{l-1}^{j_{l-1}} f, \quad (4.3)$$

where  $J'_1 \equiv \emptyset$  and, for  $l > 1$ ,  $J'_l := \{(j_1, \dots, j_{l-1}) \in \mathbb{N}^{l-1} \mid \sum_{i=1}^{l-1} i j_i = l\}$ . Since  $\mathcal{L}_1, \dots, \mathcal{L}_{n-1}$  satisfy Leibniz's rule (see Appendix), they are derivatives, and we can thus define  $n-1$  vector fields  $\xi_1, \dots, \xi_{n-1}$  by requiring that, for any  $C^1$  function  $f$ ,  $\mathcal{L}_{\xi_l} f := \mathcal{L}_l f$ . Now, if  $\Psi_\lambda$  is the knight of rank  $n-1$  generated by  $\xi_1, \dots, \xi_{n-1}$  as in (2.2), we can combine (4.1), (4.3), and (3.8) to get

$$\Phi_\lambda^*f = \Psi_\lambda^*f + \lambda^n \Delta_\lambda^{(n)}f, \quad (4.4)$$

where  $\Delta_\lambda^{(n)}f$  is  $O(\lambda^0)$ . This completes the proof.  $\square$

## 5 Gauge transformation and conclusions

In the previous sections we have presented the theory of Taylor's expansions for one-parameter families of diffeomorphisms on a manifold  $\mathcal{M}$ . Taking the simple case of a

flow as our basic element, we have first defined the notion of knights, and then shown that an arbitrary one-parameter family of diffeomorphisms can always be approximated by a family of knights of a suitable rank. We can now return to the problem stated in the introduction, of finding the relationship between the  $k$ th order perturbations of a tensor  $T_\lambda$  in two gauges  $\varphi_\lambda$  and  $\psi_\lambda$ .

Let  $n$  be the lowest order of differentiability of the objects contained in (1.1). It follows from Theorem 1 that the action of  $\Phi_\lambda$  is equivalent, up to the order  $\lambda^n$ , with the one of a knight  $\Psi_\lambda$ , constructed as in (2.2). Therefore, we can expand (1.1) using (3.8), and find,  $\forall k < n$ ,

$$\delta^k T^\psi = \sum_{l=0}^k \frac{k!}{(k-l)!} \sum_{J_l} \frac{1}{2^{j_2} \dots k! j_k j_1! \dots j_k!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_k}^{j_k} \delta^{k-l} T^\varphi, \quad (5.1)$$

where the various quantities are defined according to (1.2), and  $\delta^0 T^\varphi := T_0$ . Equation (5.1) gives a complete description of the gauge behaviour of perturbations at an arbitrary order. Among other applications, it allows one to obtain easily the conditions for the gauge invariance of perturbations to  $k$ th order; this problem has been discussed in some detail in reference [3]. Since the problem of gauge dependence is purely kinematical, (5.1) is valid not only in general relativity, but in any geometrical theory of spacetime.

Of course, our treatment can be easily generalised in several ways. For instance, it may happen that the perturbations are characterised by several parameters  $\lambda_1, \dots, \lambda_N$  [2], so that one is dealing with a  $N$ -parameter family of spacetime models  $M_{(\lambda_1, \dots, \lambda_N)}$  that differ from the background  $M_{(0, \dots, 0)}$ . Correspondingly, gauge transformations are associated to the action of a  $N$ -parameter family of diffeomorphisms  $\Phi : \mathcal{D} \rightarrow \mathcal{M}$ , where  $\mathcal{D}$  is an open subset of  $\mathbb{R}^N \times \mathcal{M}$  containing  $\{(0, \dots, 0)\} \times \mathcal{M}$ , and  $\Phi((0, \dots, 0), p) = p$ ,  $\forall p \in \mathcal{M}$ . One can then ask several questions about such an extension of the theory discussed in the present paper. However, we leave this topic for future investigations.

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## Appendix: Proof that the operators $\mathcal{L}_l$ satisfy the Leibniz rule

Since the operators  $\mathcal{L}_l$  are linear, the Leibniz rule is equivalent to the condition  $\mathcal{L}_l f^2 = 2f \mathcal{L}_l f$ , for any  $C^1$  function  $f$ . This property can be established for any  $l$  by induction.



It trivially holds for  $l = 1$ , so there exists a vector field  $\xi_1$  such that  $\mathcal{L}_1 f = \mathcal{L}_{\xi_1} f, \forall f$ . Let us suppose that this is true up to  $l - 1$ , so that there are  $l - 1$  vector fields  $\xi_1, \dots, \xi_{l-1}$  such that  $\mathcal{L}_k f = \mathcal{L}_{\xi_k} f, \forall k \leq l - 1$  and  $\forall f$ . Then we must prove that

$$2f\mathcal{L}_l f = \frac{d^l}{d\lambda^l} \Big|_0 (\Phi_\lambda^* f)^2 - \sum_{J'_l} \frac{l!}{2^{j_2} \dots (l-1)! j_{l-1}! j_1! j_2! \dots j_{l-1}!} \mathcal{L}_{\xi_1}^{j_1} \mathcal{L}_{\xi_2}^{j_2} \dots \mathcal{L}_{\xi_{l-1}}^{j_{l-1}} f^2. \quad (\text{A.1})$$

Recalling (4.3), we have

$$\begin{aligned} \frac{d^l}{d\lambda^l} \Big|_0 (\Phi_\lambda^* f)^2 &= \sum_{k=0}^l \binom{l}{k} \frac{d^k}{d\lambda^k} \Big|_0 (\Phi_\lambda^* f) \frac{d^{l-k}}{d\lambda^{l-k}} \Big|_0 (\Phi_\lambda^* f) \\ &= 2f\mathcal{L}_l f + 2f \sum_{J'_l} \frac{l!}{2^{j_2} \dots (l-1)! j_{l-1}! j_1! j_2! \dots j_{l-1}!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_{l-1}}^{j_{l-1}} f \\ &\quad + \sum_{k=1}^{l-1} \binom{l}{k} \left( \sum_{J_k} \frac{k!}{2^{j_2} \dots k! j_k! j_1! j_2! \dots j_k!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_k}^{j_k} f \right) \\ &\quad \left( \sum_{J_{l-k}} \frac{(l-k)!}{2^{j_2} \dots (l-k)! j_{l-k}! j_1! j_2! \dots j_{l-k}!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_{l-k}}^{j_{l-k}} f \right); \end{aligned} \quad (\text{A.2})$$

therefore, (A.1) is satisfied iff

$$\begin{aligned} &\sum_{J'_l} \frac{l!}{2^{j_2} \dots (l-1)! j_{l-1}! j_1! j_2! \dots j_{l-1}!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_{l-1}}^{j_{l-1}} f^2 \\ &= 2f \sum_{J'_l} \frac{l!}{2^{j_2} \dots (l-1)! j_{l-1}! j_1! j_2! \dots j_{l-1}!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_{l-1}}^{j_{l-1}} f \\ &\quad + \sum_{k=1}^{l-1} \binom{l}{k} \left( \sum_{J_k} \frac{k!}{2^{j_2} \dots k! j_k! j_1! j_2! \dots j_k!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_k}^{j_k} f \right) \\ &\quad \left( \sum_{J_{l-k}} \frac{(l-k)!}{2^{j_2} \dots (l-k)! j_{l-k}! j_1! j_2! \dots j_{l-k}!} \mathcal{L}_{\xi_1}^{j_1} \dots \mathcal{L}_{\xi_{l-k}}^{j_{l-k}} f \right), \end{aligned} \quad (\text{A.3})$$

for any  $f$  and for any choice of the vector fields  $\xi_1, \dots, \xi_{l-1}$ . This relationship could be proved by brute force. However, it is easier to follow an alternative path. Let us consider a knight  $\Psi_\lambda$  of rank  $l$ , generated by the vectors  $\xi_1, \dots, \xi_{l-1}$ , and by a new arbitrary vector  $\xi_l$ . Then one can compute

$$\frac{d^l}{d\lambda^l} \Big|_0 (\Psi_\lambda^* f)^2$$

using (3.8), from which (A.3) follows straightforwardly.

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## Figure captions

Fig. 1 The action of a knight diffeomorphism  $\Psi_\lambda$  of rank 2 generated by  $\xi_1$  and  $\xi_2$ . Solid lines: integral curves of  $\xi_1$ . Dashed lines: integral curves of  $\xi_2$ . The parameter lapse between  $p$  and  $\phi_\lambda^{(1)}(p)$  is  $\lambda$ , and that from  $\phi_\lambda^{(1)}(p)$  to  $\Psi_\lambda(p)$  is  $\lambda^2/2$ .

Fig. 2 The action of a knight diffeomorphism of rank two, represented in a chart to order  $\lambda^2$ .



